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A Note on QED with Magnetic Field and Chemical Potential

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Abstract

Using a generalized proper-time method, we obtain expressions for the fermion density and the QED effective Lagrangian for an external magnetic field at finite chemical potential. The effective Lagrangian and the density are here written in terms of elementary functions, summed over a finite number of filled Landau levels.

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The study of finite temperature and density quantum electrodynamics (QED) with a nonvanishing average magnetic field is of considerable interest as it may be associated with the electron-positron plasma in compact stellar objects (e.g. neutron stars and magnetic white dwarfs), where the fermion density and the magnitude of the magnetic field may be extremely high (see e.g. Refs. [1, 2]).

The QED thermodynamical potential at finite temperature and density with a static uniform magnetic field was calculated already 25 years ago in Ref. [3], and using a generalization of Fock–Schwinger's proper-time method for $T, \mu \neq 0$ (where μ is the chemical potential) later in Ref. [4]. The interest to this problem was renewed in Ref. [5], where an elegant generalization of Fock–Schwinger's proper-time method in the case of nonzero magnetic field and chemical potential was made. Using a real–time thermal formalism, the results of Ref. [5] were completed and generalized to finite temperature in Ref. [6]. The expressions obtained for the effective action in the above cited references were rather complicated and did include some proper-time like integrals and/or infinite sums.

Here we want to demonstrate that using the formalism, elaborated in Ref. [5] we may move forward and for the finite density QED with an external magnetic field at zero temperature show that it is possible to obtain simple expressions for the fermion density and the effective Lagrangian. The effective Lagrangian is here written in terms of elementary functions as a sum over a finite number of (partially) filled Landau levels, and agrees with the zero temperature limit of the fermion partition function. As an application we confirm that the magnetisation obtained from this effective action does exhibit the relativistic de Haas–van Alphen effect [3, 6, 7].

We shall here consider finite density QED with a nonvanishing average magnetic field. Including the chemical potential the corresponding Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - \gamma_0\mu - m)\psi \quad , \tag{1}$$

where we have chosen the gauge $A_\mu = (0, -x_2B, 0, 0)$.

In order to calculate the one-loop correction to the effective Lagrangian, $\int d^4x\mathcal{L}^{\text{eff}} = -i\ln \text{Det}(i\partial\!\!\!/ - eA\!\!\!/ - \gamma_0\mu - m)$, we shall first evaluate the fermion density $\rho = \frac{\partial\mathcal{L}^{\text{eff}}}{\partial\mu}$. We may then reconstruct the effective Lagrangian according to

$$\mathcal{L}^{\text{eff}}(B,\mu) = \mathcal{L}^{\text{eff}}(B) + \tilde{\mathcal{L}}^{\text{eff}}(B,\mu) \quad , \tag{2}$$

where

$$\tilde{\mathcal{L}}^{\text{eff}}(B,\mu) = \int_0^\mu \rho(B,\mu')d\mu' \quad , \tag{3}$$

is the contribution due to the finite density, and

$$\mathcal{L}^{\text{eff}}(B) = -\frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s^3}\left[eBs\coth(eBs) - 1 - \frac{1}{3}(eBs)^2\right]\exp(-m^2 s) \tag{4}$$

is the well known vacuum part of the effective Lagrangian in the purely magnetic case [8].

We may rewrite the expression for the fermion density as

$$\rho = i\,\mathrm{tr}\,(\gamma_0 G(x = x')) = i\int \frac{d^4p}{(2\pi)^4}\mathrm{tr}\,(\gamma_0 G(B,\mu;p)) \quad , \tag{5}$$

where the trace is over spinor indices only, and $G(x,x')$ is the fermion Green's function in configuration space. We shall use the expression obtained in Ref. [5] for the Green's function in momentum space

$$G(B,\mu;p) =$$

$$-i\theta((p_0+\mu)\mathrm{sign}p_0)\int_0^\infty ds\exp\left\{is\left[(p_0+\mu)^2 - p_\parallel^2 - p_\perp^2\frac{\tan(eBs)}{eBs} - m^2 + i\varepsilon\right]\right\}\times$$

$$\Big\{(1+\gamma_1\gamma_2\tan(eBs))\left[\gamma_3 p_\parallel - \gamma_0(p_0+\mu) - m\right] + \{1+\tan^2(eBs)\}(\gamma_1 p_1 + \gamma_2 p_2)\Big\}$$

$$+i\theta(-(p_0-\mu)\mathrm{sign}p_0)\int_0^\infty ds\exp\left\{-is\left[(p_0+\mu)^2 - p_\parallel^2 - p_\perp^2\frac{\tan(eBs)}{eBs} - m^2 - i\varepsilon\right]\right\}\times$$

$$\Big\{(1-\gamma_1\gamma_2\tan(eBs))\left[\gamma_3 p_\parallel - \gamma_0(p_0+\mu) - m\right] + \{1+\tan^2(eBs)\}(\gamma_1 p_1 + \gamma_2 p_2)\Big\} \tag{6}$$

where $p_\parallel$ and $p_\perp$ are the modulus of the momenta parallel and perpendicular to the magnetic field, respectively. A different $i\varepsilon$-prescription here arise for $|\mu| > |m|$ as the rules for passing poles in the fermion Green's function are changing [5, 9], since the Dirac sea is filled up to the energy μ.

Using Eq.(6) in Eq.(5), the identity $\theta(x) = 1 - \theta(-x)$, and performing a trivial change of variables, we get

$$\rho(B,\mu) = -\frac{1}{4\pi^4}\int d^4p\int_0^\infty ds\, p_0\exp\left\{is\left[(p_0+\mu)^2 - p_\parallel^2 - p_\perp^2\frac{\tan(eBs)}{eBs} - m^2 + i\varepsilon\right]\right\}$$

$$+\ \frac{1}{4\pi^4}\int_0^\mu p_0 dp_0\int d^3\mathbf{p}\int_0^\infty ds\left(\exp\left\{is\left[(p_0+\mu)^2 - p_\parallel^2 - p_\perp^2\frac{\tan(eBs)}{eBs} - m^2 + i\varepsilon\right]\right\} +\right.$$

$$\left. + \exp\left\{-is\left[(p_0+\mu)^2 - p_\parallel^2 - p_\perp^2\frac{\tan(eBs)}{eBs} - m^2 - i\varepsilon\right]\right\}\right) \quad , \tag{7}$$

where the first term on the right-hand side is vanishing.

Performing the momentum integration in Eq.(7), we get

$$\rho(B,\mu) = 2\Re e\left\{\frac{e^{3i\pi/4}}{8\pi^{3/2}}\int_0^\infty \frac{ds}{s^{5/2}}(eBs)\cot(eBs)\exp\{is(\mu^2-m^2+i\varepsilon)\}\right\} \tag{8}$$

that may be obtained from the expression for the effective Lagrangian in Ref. [5], keeping the $i\varepsilon$ prescription. This $i\varepsilon$ prescription tells us that the proper-time integral actually is to be performed slightly below the real axis. When closing the contour to obtain an exponentially decreasing integrand instead of an oscillating one, the poles of $\cot(eBs)$ will thus be encircled when the contour is closed in the upper half-plane for $\mu^2 > m^2$. The sum over the residues at these poles will form the vanishing temperature limit of the "oscillating" part of the effective Lagrangian, in agreement with Ref. [6].

The proper-time integral in Eq.(8), or the corresponding integral after the above-described Wick rotation, cannot be performed analytically. Instead, we shall here perform the proper-time integral in Eq.(7) before the integral over the momentum. This equation may be rewritten as [10]

$$\rho = \frac{1}{4\pi^4}\int_0^\mu p_0 dp_0 \int d^3\mathbf{p}\,(I(p)+I^*(p)) \quad , \tag{9}$$

where we have defined

$$I(p) = \int_0^\infty ds\exp\left\{is\left[p_0^2 - p_\parallel^2 - p_\perp^2\frac{\tan(eBs)}{eBs} - m^2 + i\varepsilon\right]\right\} \quad . \tag{10}$$

For $m^2 > p_0^2 - p_\parallel^2$ we may close the integration contour in the lower half plane

$$I(p) = -i\int_0^\infty ds\exp\left\{-s\left[m^2 - p_0^2 + p_\parallel^2 + p_\perp^2\frac{\tanh(eBs)}{eBs} - i\varepsilon\right]\right\} \quad . \tag{11}$$

The integral in Eq.(11) is diverging as $s\to\infty$ for $m^2 \le p_0^2 - p_\parallel^2$. Changing the variable of integration to $z = \tanh(eBs)$ $\quad (eB>0)$, and defining $Q = (m^2 + p_\parallel^2 - p_0^2 - i\varepsilon)/2eB$, we may rewrite Eq.(11) as [11]

$$I(p) = -\frac{i}{eB}\int_0^1 dz(1-z)^{-1+Q}(1+z)^{-1-Q}\exp\left\{-\frac{p_\perp^2}{eB}z\right\} \quad , \tag{12}$$

that has a singularity at $z=1$. From the left- and right-hand sides of Eq.(12) one may now extract the first k terms of the Taylor expansion of $(1+z)^{-1-Q}\exp\{-p_\perp^2 z/eB\}$ around

$z = 1$ (Cauchy method),

$$I(p) + \frac{i}{eB}\sum_{n=0}^{k}\int_0^1 dz a_n(p)(1-z)^{n+Q-1} =$$

$$-\frac{i}{eB}\left\{(1+z)^{-1-Q}\exp\left\{-\frac{p_\perp^2}{eB}z\right\} - \sum_{n=0}^{k}\int_0^1 dz a_n(p)(1-z)^n\right\}(1-z)^{-1+Q} \quad , \tag{13}$$

where we have defined

$$a_n(p) \equiv \frac{(-1)^n}{n!}\frac{d^n}{dz^n}\left[(1+z)^{-1-Q}\exp\{-p_\perp^2 z/eB\}\right]\Big|_{z=1} \quad . \tag{14}$$

Performing the trivial integration in the left-hand side of Eq.(13), under the assumption that $Q + n > 0$, we get

$$I(p) + \frac{i}{eB}\sum_{n=0}^{k}\frac{a_n(p)}{n+Q}(1-z)^{n+Q-1} =$$

$$-\frac{i}{eB}\int_0^1 dz\left\{(1+z)^{-1-Q}\exp\left\{-\frac{p_\perp^2}{eB}z\right\} - \sum_{n=0}^{k}\int_0^1 dz a_n(p)(1-z)^n\right\}(1-z)^{-1+Q} \quad . \tag{15}$$

The integral in the right-hand side of Eq.(15) is convergent in the halfplane $\Re e(Q) > -(k+1)$. Taking limit $k \to \infty$ we see that Eq.(15) is an analytical continuation of $I(p)$ on the whole complex plane $p_0^2 - p_\parallel^2$, excluding the points $p_0^2 - p_\parallel^2 = m^2 + 2eBn, \quad n = 0, 1, 2, \ldots,$ where $I(p)$ has simple poles (these poles are nothing but the familiar relativistic Landau levels).

Substituting the expression for $I(p)$ rewritten as in Eq.(15) into Eq.(9) we see that the regular parts of $I(p)$ and $I^*(p)$ cancel, and thus

$$\rho(B,\mu) = \frac{i}{2\pi^4}\int_0^\mu p_0 dp_0 \int d^3\mathbf{p}\sum_{n=0}^{\infty}\times$$

$$\left(\frac{a_n(p)}{p_0^2 - p_\parallel^2 - m^2 - 2eBn + i\varepsilon} - \frac{a_n^*(p)}{p_0^2 - p_\parallel^2 - m^2 - 2eBn - i\varepsilon}\right) \quad . \tag{16}$$

Using the identity $(x \pm i\varepsilon)^{-1} = \wp(x^{-1}) \mp i\pi\delta(x)$, we see that also the principal values cancel, and the only nonvanishing contribution comes from the poles

$$\rho(B,\mu) \quad = \quad \frac{1}{\pi^3}\int_0^\mu p_0 dp_0 \int d^3\mathbf{p}\sum_{n=0}^{\infty} a_n(p)\delta(p_0^2 - p_\parallel^2 - m^2 - 2eBn)$$

$$= \frac{1}{2\pi^2}\int_{-\infty}^{\infty} dp_{\|}\int_0^{\infty} d(p_\perp^2)\sum_{n=0}^{\infty} a_n(p_{(n)})\theta(\mu^2 - p_{\|}^2 - m^2 - 2eBn) \quad , \tag{17}$$

where $p_{(n)}$ denotes the four-momentum, such that $(p_{0(n)})^2 = m^2 + (p_{\|(n)})^2 + 2eBn$. The Heavyside step-functions here does accordingly describe the number of (partially) filled Landau levels. For example, for $0 < \mu^2 - m^2 < 2eB$, only the lowest Landau level ($n = 0$) contributes to the density

$$\begin{aligned} \rho_0(B,\mu) &= \frac{1}{2\pi^2}\int_{-\infty}^{\infty} dp_{\|}\int_0^{\infty} d(p_\perp^2)a_0(p_0)\ \theta(\mu^2 - p_{\|}^2 - m^2 - 2eBn) \\ &= \frac{eB}{2\pi^2}\sqrt{\mu^2 - m^2}\ \theta(\mu^2 - m^2) \quad . \end{aligned} \tag{18}$$

For $0 < \mu^2 - m^2 < 4eB$, we get similarly $\rho(B,\mu) = \rho_0(B,\mu) + \rho_1(B,\mu)$, where

$$\begin{aligned} \rho_1(B,\mu) &= \frac{1}{\pi^2}\sqrt{\mu^2 - m^2 - 2eB}\ \theta(\mu^2 - m^2 - 2eB)\int_0^{\infty} d(p_\perp^2)\frac{p_\perp^2}{eB}\exp\left(-\frac{p_\perp^2}{eB}\right) \\ &= \frac{eB}{\pi^2}\sqrt{\mu^2 - m^2 - 2eB}\ \theta(\mu^2 - m^2 - 2eB) \quad . \end{aligned} \tag{19}$$

The general expression for the fermion density in a static uniform magnetic field B, may thus be written as a sum over a finite number of occupied Landau levels

$$\rho(B,\mu) = \sum_{n=0}^{\left[\frac{\mu^2 - m^2}{2eB}\right]} \rho_n(B,\mu) \quad , \tag{20}$$

where $[\ldots]$ denotes the integral part. The contribution from the n-th Landau level is

$$\rho_n(B,\mu) = b_n\,\frac{eB}{2\pi^2}\sqrt{\mu^2 - m^2 - 2eBn} \quad , \tag{21}$$

where we have defined

$$b_n \equiv 2 - \delta_{n,0} \quad , \tag{22}$$

since the lowest Landau level ($n = 0$), unlike the higher levels, only contains fermions with one projection of the spin; as found in the above examples of ρ_0 and ρ_1, cf. Eq.(26).

We have thus found an expression for the fermion density in a nonvanishing average magnetic field, in terms of elementary functions in a discrete finite sum over filled Landau levels. This result may be well understood from the index theorem approach [12]. The density (fermion number) depends on the difference of numbers of filled positive and

negative energy levels, which in this case is the (semidescrete) number of Landau levels in the interval $[0,\mu]$.

In the limit of vanishing magnetic field, $eB \to 0$, the Riemann sum of Eq.(21) may be rewritten as an integral

$$\rho(\mu) = \frac{1}{2\pi^2}\int_0^{\mu^2-m^2} dx(\mu^2-m^2-x)^{1/2}\theta(\mu^2-m^2) = \frac{1}{3\pi^2}\left(\mu^2-m^2\right)^{3/2}\theta(\mu^2-m^2) \quad (23)$$

which is the familiar expresion for the fermion density.

In Figure 1 we show the density as a function of the chemical potential for fixed magnetic field, and in Figure 2 the density is given as a function of the magnetic field for fixed chemical potential. We see that the density is showing an oscillating behaviour as consecutive Landau levels are passing the Fermi level.

Integrating Eq.(20) with respect to the chemical potential, we find the part of the effective Lagrangian due to the finite density as

$$\tilde{\mathcal{L}}^{\mathrm{eff}}(B,\mu) = \sum_{n=0}^{\left[\frac{\mu^2-m^2}{2eB}\right]} \mathcal{L}_n(B,\mu) \quad , \quad (24)$$

where the contribution from the n-th Landau level is

$$\mathcal{L}_n(B,\mu) = b_n \frac{eB}{4\pi^2}\left\{\mu\sqrt{\mu^2-m^2-2eBn} - (m^2+2eBn)\ln\left(\frac{\mu+\sqrt{\mu^2-m^2-2eBn}}{\sqrt{m^2+2eBn}}\right)\right\} \quad . \quad (25)$$

Here we have used the zero temperature proper-time method to calculate the fermion density and effective Lagrangian. It is noteworthy to compare with the approach of quantum statistical mechanics. By comparing the generating functional for fermionic Green's functions in imaginary time [13], with the partition function in the grand canonical ensemble (Z), we find that $\tilde{\mathcal{L}}^{\mathrm{eff}} = \frac{1}{\beta V}\ln Z$. The relativistic fermion energy-levels in a static uniform magnetic field are found as [14]

$$E_{k,\lambda}(p_\parallel) = \sqrt{m^2+p_\parallel^2+2eB(k+\lambda-1)} \quad , \quad (26)$$

where $k = 0,1,2,\ldots$ corresponds to the quantized orbital angular momentum, and $\lambda = 1,2$ describes the projection of spin. Using the ordinary relativistic dispersion law to reintroduce the momenta orhogonal to the magnetic field, we find the density of states $V\, eB/(2\pi)^2$, and readily obtain

$$\tilde{\mathcal{L}}^{\mathrm{eff}}(B,\mu,T) =$$

$$\frac{1}{\beta}\frac{eB}{(2\pi)^2}\sum_{k=0}^{\infty}\sum_{\lambda=1}^{2}\int_{-\infty}^{\infty} dp_\parallel \left\{\ln[1+e^{-\beta(E_{k,\lambda}(p_\parallel)-\mu)}] + \ln[1+e^{-\beta(E_{k,\lambda}(p_\parallel)+\mu)}]\right\} \quad . \quad (27)$$

Integrating by parts with respect to $p_\parallel$ in Eq.(27) we find

$$\tilde{\mathcal{L}}^{\rm eff}(B,\mu,T) = \frac{eB}{(2\pi)^2}\sum_{k=0}^{\infty}\sum_{\lambda=1}^{2}\int_{-\infty}^{\infty} dp_\parallel \frac{p_\parallel^2}{E_{k,\lambda}(p_\parallel)}\left\{\frac{1}{1+e^{\beta(E_{k,\lambda}(p_\parallel)-\mu)}}+\frac{1}{1+e^{\beta(E_{k,\lambda}(p_\parallel)+\mu)}}\right\} \quad . \tag{28}$$

Using $1/(1+e^{\beta(E_{k,\lambda}(p_\parallel)\mp\mu)}) \to \theta[\pm\mu - E_{k,\lambda}(p_\parallel)]$, as $\beta\to\infty$, we may in the limit of vanishing temperature perform the momentum integral and arrive at Eq.(24).

The magnetisation of the fermion gas is now easily found by performing the derivative with respect to the magnetic field, $\tilde{M} = \frac{\partial}{\partial B}\tilde{\mathcal{L}}^{\rm eff}$, with the result

$$\tilde{M}(B,\mu) = \sum_{n=0}^{\left[\frac{\mu^2-m^2}{2eB}\right]} M_n(B,\mu) \quad , \tag{29}$$

where the contribution from the n-th Landau level is

$$M_n(B,\mu) = b_n\,\frac{e}{4\pi^2}\left\{\mu\sqrt{\mu^2-m^2-2eBn} - (m^2+4eBn)\ln\left(\frac{\mu+\sqrt{\mu^2-m^2-2eBn}}{\sqrt{m^2+2eBn}}\right)\right\} \quad . \tag{30}$$

We notice that $\lim_{(\mu^2-m^2-2eBn\to 0^+)}\mathcal{L}_n(B,\mu)=0$, and $\lim_{(\mu^2-m^2-2eBn\to 0^+)}M_n(B,\mu)=0$, so that the Lagrangian density as well as the magnetisation are continuous. Figure 3 shows the total magnetisation, $M^{\rm tot}(B,\mu)\equiv \frac{\partial}{\partial B}\mathcal{L}^{\rm eff}(B,\mu)$ [6], as a function of the magnetic field for fixed chemical potential (however, the vacuum contribution $\frac{\partial}{\partial B}\mathcal{L}^{\rm eff}(B)$ is small in this range of parameters). In agreement with Refs. [3, 6, 7], we see that for low temperatures (here $T=0$), the relativistic fermion gas exhibits the de Haas–van Alphen effect.

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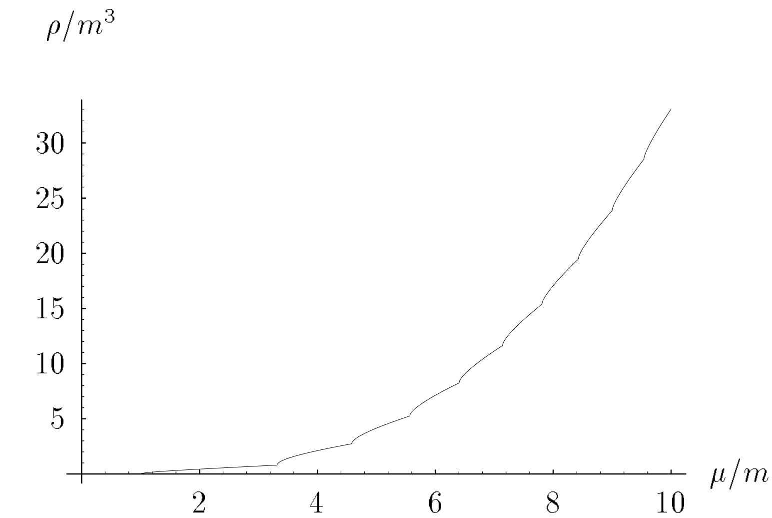

Figure 1: *The fermion density as a function of the chemical potential for fixed magnetic field,* $eB/m^2 = 5$.

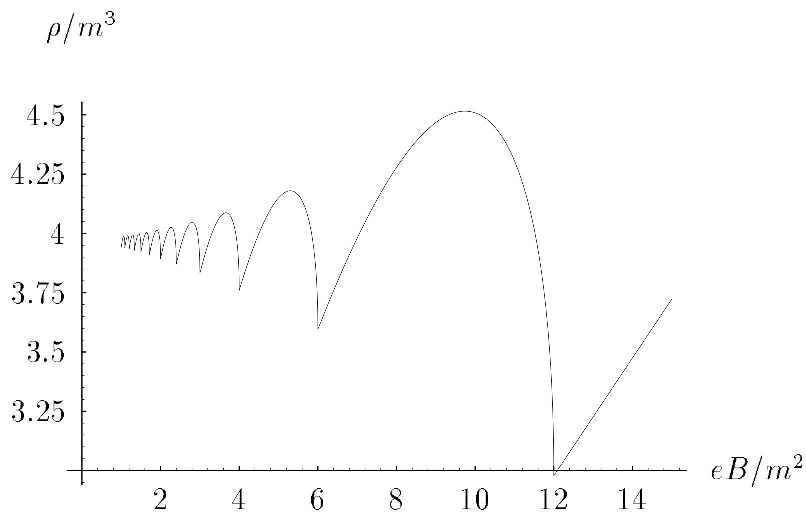

Figure 2: *The fermion density as a function of the magnetic field for fixed chemical potential,* $\mu = 5$.

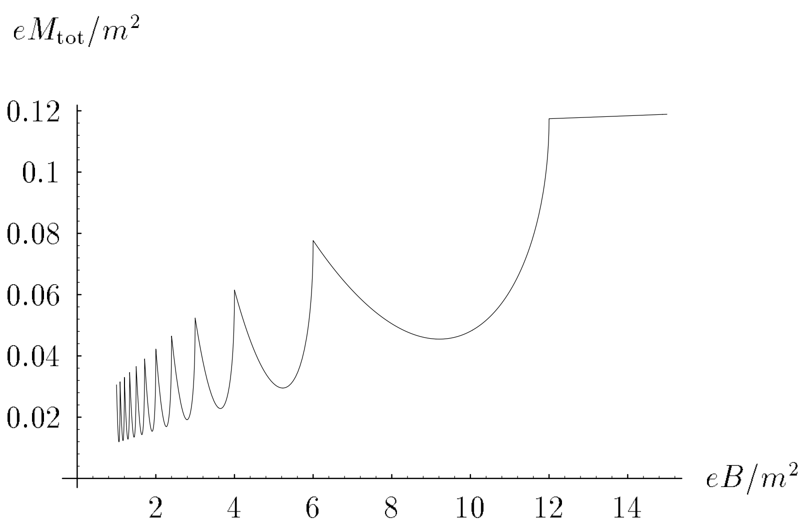

Figure 3: *The total magnetisation as a function of the magnetic field for fixed chemical potential,* $\mu = 5$.